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CHALLENGING MATHEMATICS BY "ARCHIMEDES"

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Abstract. This paper presents the activities of the Serbian mathematical club "Archimedes" aiming at challenging mathematics in and beyond the classroom. It also summarizes main achievements of the club throughout three decades of its existence, focusing on the main findings from its rich experience in challenging mathematics. Examples of challenging tasks with some didactical remarks are given in the appendices.

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What is "Archimedes"?

There is an unusual organization in Belgrade, Serbia, called the "Archimedes" Mathematical Club. "Archimedes" is much more than some club, not only because of its diverse activities, but also because of its respectable results concerning further mathematical education of students and professional development of mathematics teachers. It is particularly interesting and pleasant to mention that, for more than thirty years, "Archimedes" has dealt with most questions raised by the ICMI Study 16 Discussion Document [1], particularly when challenging mathematics beyond the classroom is in focus. Such a state is easy to verify if one analysis the activities of the Club. Main facts about the Club are summarized in Exhibit 1.

Many of "Archimedes" students are today successful researchers or university professors (unfortunately mainly abroad). Among them are Dr Ranko Lazić (UK), Dr Marko Stošić (Portugal) and Dr Đorđe Milićević (USA), to mention just a few. In the last six years, several students from the "Archimedes" Olympic Group have been awarded the title "Student of Generation" at the Faculty of Mathematics, Belgrade University: Đorđe Milićević, Marko Stošić, Ivan Matić, Vladimir Lazić, Miljan Brakočević and Tatjana Simčević. From 1992 to 2005 students from Serbia who participated in the International Mathematical Olympiads (under the flag of Yugoslavia or Serbia & Montenegro) received 45 medals (prizes), 42 of which were awarded to students involved in the "Archimedes" school (usually for several years with some of them even from early grades). For the Balkan Mathematical Olympiads arranged from 1995 to 2005, these figures were 39 and 37, respectively. In 2005, at the national mathematical contest, among the thirteen best students in

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Dedicated to the 16th ICMI Study Challenging Mathematics in and beyond the Classroom, Trondheim, Norway, 27 June–3 July 2006. All estimations concerning the achievements of the "Archimedes" Mathematical Club have been made by the first author.

grade 9, ten were from the same "Archimedes" mathematical class. Of course, credits for such outstanding performances of these students not only go to "Archimedes" but also to their regular secondary schools (in most cases to Mathematical Gymnasium of Belgrade—a specialized high school for students talented in mathematics), and to the Mathematical Society of Serbia as the organizer of all competitions at our national level and of the participation of Yugoslav, resp. Serbia & Montenegro teams in the international competitions.

Club founded in 1973. *Main goals*: (1) to organize mathematical schools and summer and winter mathematical camps; (2) to popularize mathematics and science in general; (3) to arrange mathematical competitions; (4) to deal with professional development of mathematics teachers; (5) to build up a specialized library containing some of the leading national and international mathematical publications; and (6) to publish various materials (tests, books, booklets, magazines, etc.) supporting the activities of the club. *Members*: pupils and students of mathematics at all educational levels; teachers of mathematics and informatics at all educational levels; mathematical enthusiasts and other interested adults. Club governed by an 11-member board. *Instructors*: distinguished teachers from primary and secondary schools as well as experts from universities and other institutions (mostly enthusiasts).

Basic statistics:

- more than 20,000 students in mathematical schools and about 10,000 students in 90 mathematical camps;
- 70 lectures/presentations on the popularization of mathematics and science and about 200 other gatherings (quizzes, exhibitions, etc.) with about 22,000 persons present;
- 60 mathematical competitions with about 3,200 teams (one team per school) comprising 4 or 5 students;
- about 1,300 presentations/lectures for teachers of mathematics and informatics in primary and secondary schools (at seminars and other professional events) with about 50,000 teachers present;
- 25,000 books and about 5,000 issues of journals and magazines;
- about 300 titles of various publications issued in more than two million copies; and
- about 26,000 registered members (more than 90% are pupils and students).

`www.arhimedes.co.yu`

Exhibit 1. "*Archimedes*" *business card*

One of the founders of the club and the main person in charge of all its activities is Bogoljub Marinković, the second author of this paper, who worked as educational adviser for mathematics at Belgrade Educational Board, Serbian Educational Board, and Serbian Ministry of Education for more than thirty years. For his highly dedicated work in the field of mathematical competitions at the national level, he received "Paul Erdös" award for 2002, which was presented to him by Professor Peter Taylor, President of WFNMC, at a special ceremony at the "Arhimedes" premises in Belgrade in August 2003 (see `www.amt.canberra.edu.au/erdmarin.html`).

Despite its successful activities (continuously and systematically arranged all these years), the financial support from the state and other local authorities has been minimal. Only the fact that many students who have been trained in the schools and camps of "Archimedes" have, as mentioned above, traditionally been among the best solvers at local examinations and regional, state and international

Peter Taylor presents "Paul Erdös" award to Bogoljub Marinović

mathematical contests recommends the club for a wider support of and a stronger donation from public and governmental agencies.

What does the experience of "Archimedes" mainly tell us?

Strategic issues

1. *Only a continuous and well-planed use of challenges gives good results.* In the "Archimedes" schools students in each grade have 25–30 meetings (90-minute lessons) given throughout school year (one lesson per week). Two programmes for deepening and extending mathematical knowledge are applied: a standard one (for students who like mathematics but are not competition-minded) and an advanced one (for students with a good record at mathematical contests who have passed the "Archimedes" entrance examination). At the end of each programme, a test is given to all students. Because students work in homogenous groups taught by distinguished teachers (many of whom have developed their expertise within the club), the test results are very good. Note that, because a continuous and well-planed work with candidates for national teams for international mathematical competitions is also applied (our Olympic Group), students who develop their mathematical knowledge and skills at "Archimedes" are, as already mentioned, the most successful members of these teams.

2. *Mathematical competitions should require both basic and advanced mathematical knowledge.* A good combination of team and individual competitions at a national level is achieved by annual "Archimedes" mathematical contests (open team championships of Serbian secondary schools in two groups: grades 4–8 and grades 9–12). A distinguished feature of these competitions is that students in each grade work on two groups of tasks (see [3]): one group comprises tasks from the official mathematics curriculum, whereas the other consists of non-standard tasks from the additional mathematical curriculum aiming at students with special interest in mathematics (these are usually students particularly talented for mathematics). Our position is that challenges given at competitions should not ignore the official mathematics curriculum. In 2006 "Archimedes" organized a Kangaroo-like competition with about 7,000 participants from 130 primary and secondary schools across Serbia. Bearing in mind somewhat inadequate design of this competition (it focuses on two consecutive grades not leaving enough space for tasks covered by the official curriculum), this competitions was redesigned in the spirit of the above-mentioned annual competitions. Such a redesigned competition, called "Mislisha" ("Young thinker" in Serbian), was a success indeed.

3. *Even a continuous and well-planed traditional professional development of mathematics teachers has a small global impact on the use of challenges in the classroom.* As the ICMI Study 16 Discussion Document [1] underlines, providing students with mathematically challenging situations is itself a challenge for mathematics teachers who need an improved pre-service and in-service professional development. Because faculties in Serbia have not offered an adequate pre-service professional development for the use of mathematical challenges, "Archimedes" has organized a continuous and well-planed in-service professional development of mathematics teachers (ten lectures/presentations per year; one per month) and traditional winter seminars (with 6–8 lectures/presentations). However, the impact of these traditional forms of professional development on the use of challenges in the classroom has been relatively small. Although a precise estimate of this impact is not available, the results from "Archimedes" mathematical competitions (recall that there are two types of tasks) suggest that, at the global level, just about 5-10% of all students have met mathematical challenges in the way cultivated in the club. Because of that, "Archimedes" also needs to develop and maintain other forms of continuous in-service professional development, e.g., a Web-based one. These forms should, among other things, not only present activities whereby challenges may be introduced and utilized in and beyond the classroom, but also examine factors that may influence the outcomes of the use of challenges (see [2]). Of course, to achieve a better professional development in Serbia in general, we need to have detailed guidelines, more encouragements, and stronger requirements from the Serbian Ministry of Education.

Tactical issues

1. *Questions for mathematical quizzes should be used in regular mathematics education.* Mathematical quizzes represent a particular form of competition (on an individual or a team basis) contributing to the popularization of mathematics and the growth of motivation for learning mathematics. Questions for quizzes, usually solvable in 10–30 seconds, require a prompt and meticulous thinking, contributing to the development of mathematical reasoning as well. By using the same type of tasks in their day-to-day work, teachers of mathematics can refresh regular mathematics education for the cognitive and affective benefits of their students. As a part of its various activities (summer and winter mathematical camps, mathematical performances, Quiz "Sharp Your Mind" on a regional broadcasting station with seven episodes, etc.), "Archimedes" has arranged more than hundred quizzes so far (usually with teams of 3-5 students from different grades), which have been appreciatively received by both the competitors and the audience. A sample of questions used in the "Archimedes" quizzes is given in Appendix I.

2. *If not resolved adequately through creating and using sets of contextually different yet mathematically isomorphic challenging problems, the reuse of challenging tasks is an important obstacle to their infusion in mathematics education in and beyond the classroom.* An adequate resolution (an essential yet neglected issue so far in research on challenging mathematics) may deal with two kinds of tasks: one with mathematical grounds that are very similar or isomorphic, and the other with solution methods that are very similar or identical. Although the development of such tasks (or problems; many authors prefer to use word "problems" instead "tasks", especially when challenging mathematics is considered) is a complex enterprise, things get easier when some experience in the "dressing" of desired theoretical facts or solution methods is gained. A sample of challenging tasks for lower secondary education (grades 7–9) developed along the suggested lines is given in Appendices II and III. While the tasks presented in Appendix II share the same theoretical grounds, the tasks examined in Appendix III are solvable by the same method. Although some of them are not typical tasks from the "Archimedes" task treasure (we focus here our attention on the reuse of challenging tasks not on the content of this treasure), they do have many of the main features of problems used in the "Archimedes" activities. These features, which have served as a valuable framework for developing or selecting challenging tasks in "Archimedes", can be summarized as follows: (1) there is a good mathematical idea behind the task; (2) the task is not routine; (3) the task is interesting with respect to formulation and content; (4) the task has a nice and perhaps unexpected solution(s); (5) the task requires its solver to stretch his/her mind; and (6) the solution of the task is usually short and not complicated, enabling the solver to use knowledge and skills traditionally learned in the classroom.

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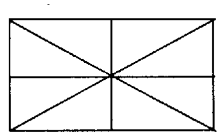
Appendix I – Mathematical quizzes

Sample of questions for a quiz of cleverness for pupils in primary education

1. Write down (express) 0 by using 3 three times. (10 seconds to come up with an answer)
   Answer: $(3-3):3$ or $(3-3)\cdot 3$.
2. Make seven out of four sticks. (10 seconds)
   Answer (Roman number is applied): I I I I → VII.
3. A string is to be cut in 5 pieces. How many cuts are needed? (10 seconds)
   Answer: 4.
4. Mouse's tail equals 12 cm and one third of the tail. How long is that tail? (15 seconds)
   Answer: $12\,\text{cm} + 6\,\text{cm} = 18\,\text{cm}$.
5. Steve and John had the same number of marbles. Steve gave 3 marbles to John. John then had more marbles than Steve. How many more? (10 seconds)
   Answer: 6.

Sample of questions for a quiz of cleverness for pupils in lower secondary education

1. Take two numbers among -9, -8, -7, 4, 5, 6 so that their product is a) the smallest; b) the largest. What is that product? (10 seconds)
   Answer: a) $(-9)\cdot(+6) = -54$; b) $(-9)\cdot(-8) = 72$.
2. Transpose one stick in VI + V = IX to achieve correct equalities. (20 seconds)
   Answer (Roman numbers are used): V+IV = IX or IV+V = IX or VI+IV = X or VI + V = XI.
3. One twelfth and a half of a number is 10. What is that number? (20 seconds)
   Answer: 80.
4. There are blue, green and red balls in a box (in sufficient numbers for each colour). How many balls should be minimally taken from the box in a blind draw so that at least four balls are of the same colour? (20 seconds)
   Answer: 10.
5. How many a) segments; b) triangles can be found in the picture below. (30 seconds)



   Answer: a) 24; b) 16.

6. Write down 100 (an expression equals to 100) by using a) 5 six times; b) 3 seven times. (30 seconds)
   Answer: a) $55 + 55 - 5 - 5$; b) $333 : 3 - 33 : 3$ (there are other solutions as well).
7. Come up with a sensible word by using letters I, R, L, G, O, M, K, A. (45 seconds)
   Answer: KILOGRAM.

Appendix II – A sample of challenging tasks with mathematically isomorphic grounds

1. Two markers cost more than three pencils. Do 5 markers cost more than 7 pencils (no discount offered)? [yes; $2x > 3y$, $x > 1.5y$, $5x > 7.5y > 7y$]
2. Marabs and Sarabs live in one county. Each Marab knowns 7 Sarabs and 9 Marabs, whereas each Sarab knows 8 Sarabs and 6 Marabs. Are there more Sarabs than Marabs? [no; make, for example, use of the fact that the number of letters sent from Marabs to known Sarabs is equal to the number of letters Sarabs received from Marabs; $7m = 6s$; $m = 6/7\,s$; $m < s$; this problem is a kind of conundrum where the given data have no effect on the solution process]
3. Two car garages in a town are competing for customers. At present, more cars can be parked in the garage with two stories than in another one with three stories. Because of that, the garage with three stories will be extended to five stories. Will the extension of the other garage from two stories to three stories enable it to continue to have more parking space? [not necessarily; $2x > 3y$, $x > 1.5y$, $3x > 4.5y < 5y$]
4. The circumference of an equilateral triangle is greater than that of a rhombus. Three these triangles are used to form an equilateral trapezium, whereas two those rhombus are combined to form an arrow shaped figure. Which of these two formed figures has a greater circumference? [trapezium; $3x > 4y$ implies $5x > 6y$]
5. At a mathematical contest each solver correctly solved at least five tasks. The jury observed a curious fact: each task was correctly solved exactly by four students. Were there more solvers or tasks to be solved at this contest? [tasks; $5s < 4t$, $1.25\,s < t$]
6. Two mountain hikers Jim and Jill climbed the same trail in the same time. Jim walked two times more than Jill rested, whereas Jill walked three times more than Jim rested. Who walked faster? [Jill; x – Jim's resting time, y – Jill's resting time; $2y + x = 3x + y$, $y = 2x$, $y > x$]
7. While walking from A to B, a group of hikers walked upwards two times longer than downwards. On their trip back, from B to A, lasting two times longer than that from A to B, the group walked downwards three times longer than upwards. Which did take longer: going upwards from A to B or B to A? [from A to B; $x_1 = 2y_1$, $y_2 = 3x_2$, $2(x_1 + y_1) = x_2 + y_2$; $x_1 = 4/3\,x_2 > x_2$]

The mathematics underlying the tasks above can be summarized by the following simple questions:

1. Suppose that $3x > 4y$ for some $x, y > 0$. Is $7x$ greater than $9y$? [yes; $x > 4/3\,y$, $7x > 28/3\,y > 9y$]

2. Assume that $4x = 3y$ for some $x, y > 0$. Is x greater than y? [no; $x = 0.75\,y < y$]

The given tasks evidence that even a simple piece of mathematics can be "dressed" in the forms of various problems, not easily perceived to be mathematically isomorphic (especially at first sight).

Appendix III – A sample of challenging tasks solvable by the same method

1. Find the area of "curviliner rectangle" $ABDC$ presented in Fig. 1 that is formed by two semicircles whose diameters are equal to 4 and two segments AB and CD of length 1. [4; if semicircle AC is translated by vector onto semicircle BD (Fig. 2), the area in question is equal to the area of rectangle $ABDC$]

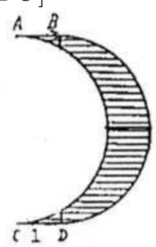


Fig. 1. Curvilinear rectangle

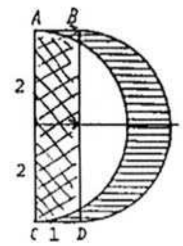


Fig. 2. Solution by translation

2. Find the area of the shaded plane form sketched in Fig. 3 if segment AB of length 24, which touches the smaller semicircle, is parallel to CD—the diameter of the bigger semicircle. [72π; the solution can easily be obtained from Fig. 4; the key step is to slide the smaller semicircle along CD until the centers of the two semicircles coincide]

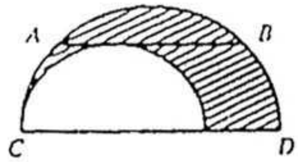


Fig. 3. Curvilinear triangle

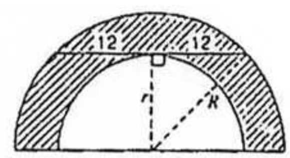


Fig. 4. Circular semi-ring

3. The bases of two isosceles triangles are collinear. Cut these triangles by a line parallel to the bases so that the segments of this line that are inside the two triangles are equal. [Fig. 5; the key step is to slide one triangle along the line of these bases until the midpoints of the bases coincide]

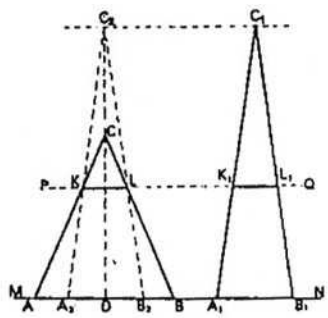


Fig. 5. Construction by translation

A general method of solution used in these three problems can be summarized in the following way: Move a figure to a position enabling a solution that can be found in a more or less straightforward way. Of course, the applied transformation is not bounded to translation. It may be other isometric transformation. A good example of using rotation can be found in the task where, for a point, a line and a circle (given objects), one has to construct an equilateral triangle having the given point as one of its vertices, whereas the two other vertices belong to the given line and circle (see Fig. 6). Furthermore, this general method can be applied to a part of a figure like in the problem of finding the area of a equilateral curvilinear triangle formed by the three arcs of the unit radius whose central angles are 60° (see Fig. 7).

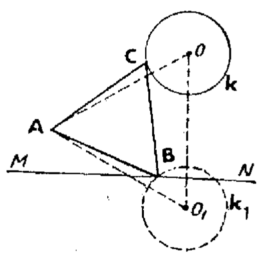


Fig. 6. Intersect the given line with the rotated circle

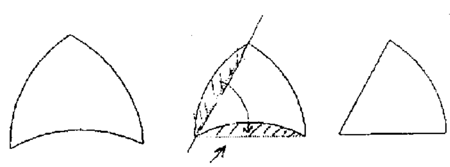


Fig. 7. From a equilater curvilinear triangle to a sector

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